

Penman-Monteith Equation Derivation

QUICK ANSWER ET008

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Introduction

These notes contain two methods for determining the Penman-Monteith equation. The first is the flux gradient and the second is a psychrometric derivation. The flux gradient method is simpler to derive, but the psychrometric method is more conceptually understandable. In the end, a discussion of weather effects on evaporation from wet surfaces and other equations are discussed.

Flux Gradient Derivation

Evaporation rate from a wet surface is determined using the flux gradient approach for estimating sensible and latent heat flux density. Recall that sensible heat flux density is estimated as

$$H = -\frac{\rho C_p}{r_H} (T - T_o) \quad (1)$$

where T is the air temperature, T_o is the surface temperature, and r_H is the resistance to sensible heat transfer. The negative sign makes H positive away from the surface. Latent heat flux density is

$$\lambda E = \frac{\rho C_p}{\gamma^* r_H} [e_s(T_o) - e] \quad (2)$$

where γ^* is the resistance to sensible heat flux and T_o is the surface temperature. The variable γ^* is the psychrometric constant that is corrected for the ratio of the resistance vapor transfer over the resistance to sensible heat transfer ($\gamma^* = \gamma \left(\frac{r_v}{r_H} \right)$). We

want to express λE in terms of parameters that we know and we do not know the surface temperature (T_o). However, we do know the air temperature (T), so we can estimate T_o using a psychrometric approximation. We need to assume that the surface temperature is approximately equal to the wet-bulb temperature or the approximation is not valid. We know that the slope Δ of the saturation vapor pressure curve evaluated at the air temperature is approximately

$$\Delta \approx \frac{e_s(T) - e_s(T_w)}{T - T_w} \quad (3)$$

Assuming that $T_w \gg T_o$, then

$$e_s(T_o) \approx e_s(T) - \Delta(T - T_o) \quad (4)$$

Substituting for $e_s(T_o)$ in Eq. 2, we get

$$\lambda E = \left(\frac{\rho C_p}{\gamma^* r_H} \right) [e_s(T) - \Delta(T - T_o) - e] \quad (5)$$

However, recall that H is a function of $T - T_o$. Rearranging Eq. 1, we can express $T - T_o$ as:

$$T - T_o = \frac{-H r_H}{\rho C_p} \quad (6)$$

Substituting this expression into Eq. 4, we get

$$\lambda E = \left(\frac{\rho C_p}{\gamma^* r_H} \right) \left[e_s(T) - \Delta \left(\frac{-H r_H}{\rho C_p} \right) - e \right] \quad (7)$$

Rearranging terms, we get

$$\lambda E = \left(\frac{\rho C_p}{\gamma^* r_H} \right) [e_s(T) - e] + \frac{\Delta}{\gamma^*} H \quad (8)$$

For a non-adiabatic process, where Q_n is the net source of external energy flux, $H = Q_n - \lambda E$. For a plant canopy, $Q_n = R_n - G$. By substitution, we get

$$\lambda E = \left(\frac{\rho C_p}{\gamma^* r_H} \right) [e_s(T) - e] + \frac{\Delta}{\gamma^*} (Q_n - \lambda E) \quad (9)$$

or

$$\lambda E = \left(\frac{\rho C_p}{\gamma^* r_H} \right) [e_s(T) - e] + \frac{\Delta}{\gamma^*} Q_n - \frac{\Delta}{\gamma^*} \lambda E \quad (10)$$

Moving λE to one side of the equation, we get

$$\left(1 + \frac{\Delta}{\gamma^*} \right) \lambda E = \left(\frac{\rho C_p}{\gamma^* r_H} \right) [e_s(T) - e] + \frac{\Delta}{\gamma^*} Q_n \quad (11)$$

but

$$\left(1 + \frac{\Delta}{\gamma^*} \right) \lambda E = \left(\frac{\Delta + \gamma^*}{\gamma^*} \right) \lambda E \quad (12)$$

so λE is

$$\lambda E = \frac{\Delta Q_n + \rho C_p [e_s(T) - e] r_H^{-1}}{\Delta + \gamma^*} \quad (13)$$

This is the Penman-Monteith equation for evaporation from a wet surface. Note that the left-hand term in the numerator is the diabatic contribution and the right-hand term is the adiabatic contribution to evaporation from a wet surface.

LE for an amphistomatous leaf (stomata on one surface)

Recall that the resistances to latent and sensible heat transfer from a wet surface are different and a correction for the psychrometric constant is

$$\gamma^* = \left(\frac{r_V}{r_H} \right) \gamma \quad (14)$$

Ignoring the small transfer through the cuticle, the resistance to vapor transfer for a leaf can be separated into the stomatal resistance to transfer from the cell surfaces, inside the leaf, to the stomata openings (r_s) and the aerodynamic resistance from the leaf surface through the boundary layer (r_b). The resistance is in series, so we have . If we assume that the aerodynamic resistance to vapor transfer through the leaf boundary layer is equal to the aerodynamic resistance to sensible heat transfer , then we have:

$$r^* = \left(\frac{r_b + r_s}{r_H} \right) \gamma = \left(\frac{r_b + r_s}{r_b} \right) \gamma = \left(1 + \frac{r_s}{r_b} \right) \gamma \quad (15)$$

Then Eq. 13 can be written as

$$\lambda E = \frac{\Delta Q_n + \rho C_p [e_s(T) - e] r_b^{-1}}{\Delta + \left(1 + \frac{r_s}{r_b} \right) \gamma} \quad (16)$$

Equation 16 is the Penman-Monteith equation for a leaf.

LE from a canopy

For a canopy, the resistance to vapor transfer from the canopy to the ambient air above can be separated into aerodynamic resistance from a fictitious level in the canopy to the air above (r_w) and the canopy resistance (r_c) from the canopy elements to the fictitious level. Typically, the fictitious level is taken to be at the level where momentum transfer equals zero ($d + z_o$ or the height of the zero plane displacement plus the roughness length). For a dense, uniform canopy, r_c is mainly affected by stomatal resistance of the plant leaves; however, some water vapor flux can also come from the soil or surface water if the plants are wet. Using the same analogy as for an amphistomatous leaf, the total resistance to vapor transfer (r_V) from the canopy is $r_V = r_w + r_c$. If we assume that the aerodynamic resistance values for sensible and latent heat flux from the fictitious level to the ambient air are equal ($r_a = r_H = r_w$), then

$$r^* = \left(\frac{r_w + r_c}{r_H} \right) \gamma = \left(\frac{r_a + r_c}{r_a} \right) \gamma = \left(1 + \frac{r_c}{r_a} \right) \gamma \quad (17)$$

If an independent measure of evaporation is available, the evaporation is not reduced by water stress, soil evaporation is negligible, and the aerodynamic resistance is known, then the Penman-Monteith equation:

$$\lambda E = \frac{\Delta Q_n + \rho C_p [e_s(T) - e] r_a^{-1}}{\Delta + \gamma \left(1 + \frac{r_e}{r_a} \right)} \quad (18)$$

provides a method to evaluate differences in plant stomatal effect on evaporation from canopies.

Psychrometric Derivation

The difference in the enthalpy (total energy content) of the ambient air and air at a wet surface occurs as a result of energy transfer to the surface that affects the sensible and latent heat content of the air. If the rate of adiabatic and diabatic heat transfer is known, then psychrometric relationships can be used to estimate the latent heat flux density. Using a psychrometric chart (Fig. 1), the rate of energy supply to increase the vapor pressure from e at point A to $e_s(T_o)$ at point C determines the total latent heat flux density (LE). From point A to B is an adiabatic process, where the energy to increase the vapor pressure from e to $e_s(T_w)$ comes from sensible heat transfer from the ambient air. The rate at which this occurs depends on the aerodynamic resistance to sensible heat transfer (r_H) from the air. From point B to point C is a diabatic process, where the rate of energy supply depends on the net external energy supply (Q_n) to the surface by radiation and conduction.

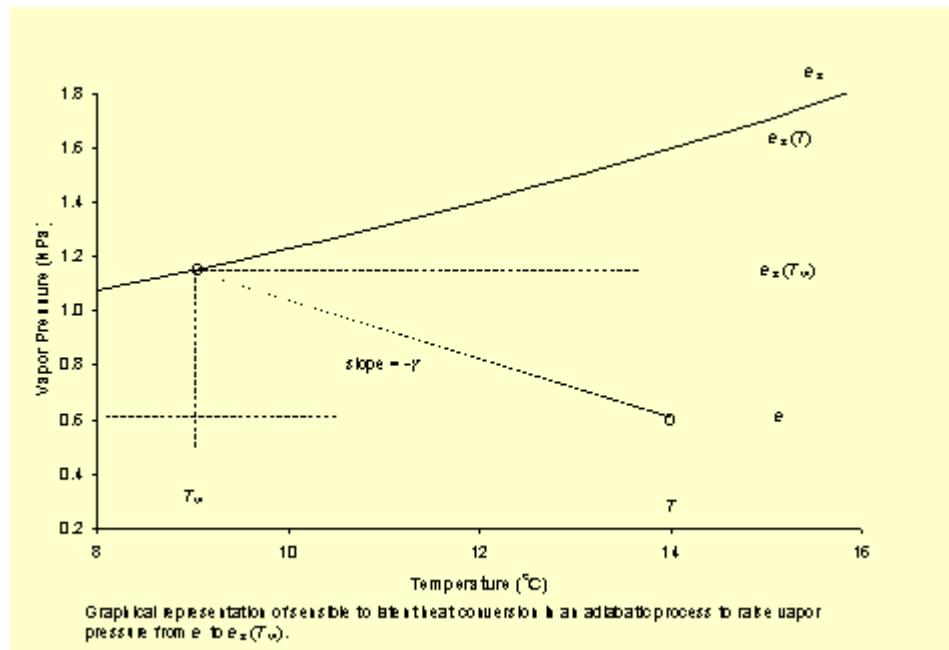


Figure 1. Adiabatic process to raise vapor pressure from e to $e_s(T_w)$

Based on the psychrometric relationships shown in Fig. 2, it is clear that

$$e_s(T) - e_s(T_w) = (T - T_w)D \quad (19)$$

and

$$e_s(T_w) - e = (T - T_w) g^* \quad (20)$$

where $\gamma^* = \left(\frac{r_v}{r_H} \right) \gamma$ accounts for the difference in resistance to sensible and latent heat flux.

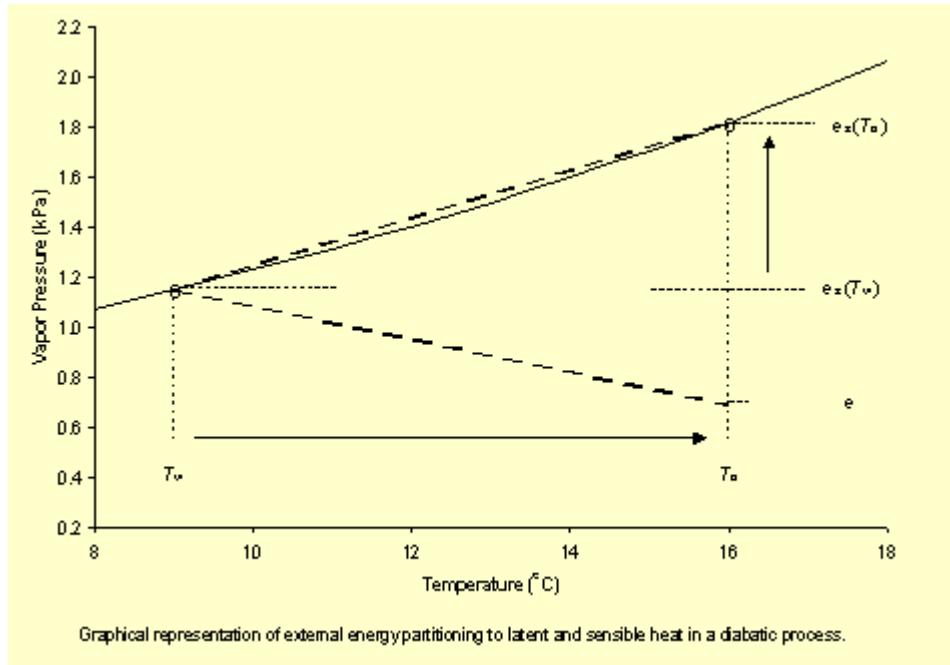


Figure 2. Energy partitioning in a diabatic process

Because

$$e_s(T) - e = [e_s(T) - e_s(T_w)] + [e_s(T_w) - e] = (T - T_w)\Delta + (T - T_w)\gamma^* \quad (21)$$

we know that

$$e_s(T) - e = (T - T_w)(\Delta + \gamma^*) \quad (22)$$

and by rearranging, we get

$$T - T_w = \frac{e_s(T) - e}{\Delta + \gamma^*} \quad (23)$$

Since the energy required to increase vapor pressure from e to $e_s(T_w)$ is equivalent to raising the temperature from T_w to T and the rate of sensible heat flux density is

$$H = -\frac{\rho C_p}{r_H} (T - T_w) \quad (24)$$

By substitution, the latent heat flux density due to adiabatic heat transfer (lE_a) is

$$\lambda E_a = \frac{\rho C_p}{r_H} \left(\frac{e_s(T) - e}{\Delta + \gamma^*} \right) \quad (25)$$

For a diabatic process [e.g., increasing vapor pressure from $e_s(T_w)$ to $e_s(T_o)$ along the saturation vapor pressure curve in Fig. 3], the air temperature must also be increased from T_w to T_o .

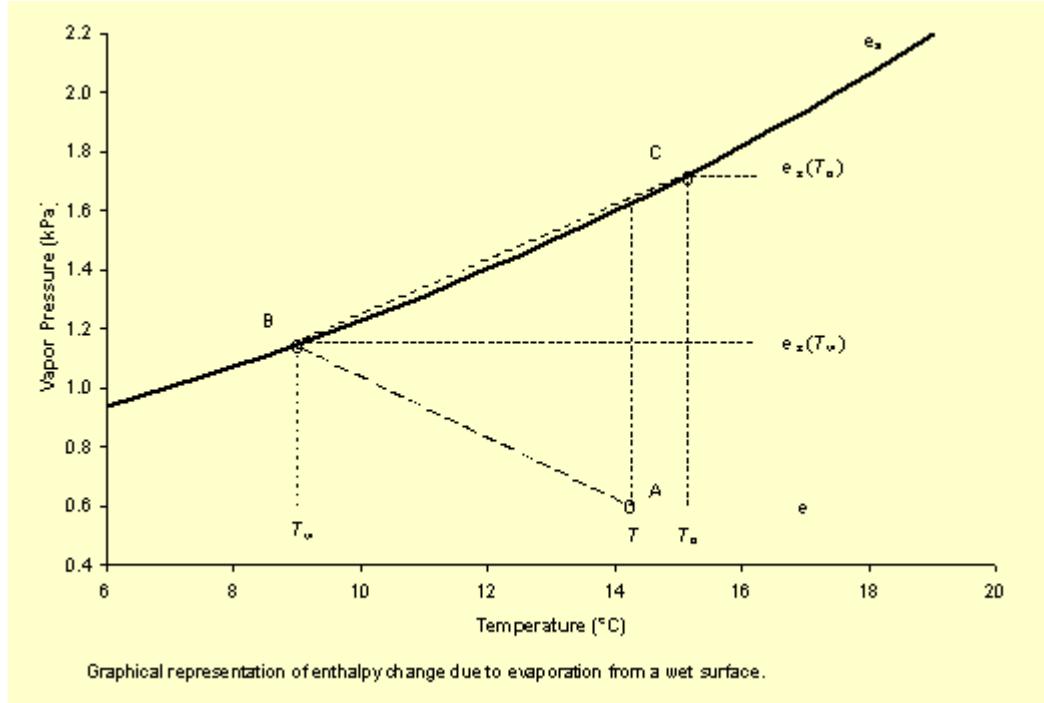


Figure 3. Enthalpy change due to heating and evaporation from a wet surface.

Part of the external energy supply contributes to evaporation and the remainder increases temperature. The amount of energy needed to increase the temperature from T_w to T_o is equal to the amount needed to increase the vapor pressure from e to $e_s(T_w)$. The total amount of energy needed to increase the temperature from T_w to T_o and the vapor pressure $e_s(T_w)$ to $e_s(T_o)$ is equal to the energy needed to raise the vapor pressure from e to $e_s(T_o)$. Therefore, the fraction of the total energy going to vaporization is given by

$$\frac{[e_s(T_o) - e_s(T_w)]}{[e_s(T_o) - e_s(T_w)] + [e_s(T_w) - e]} \quad (26)$$

Dividing the numerator and denominator by $(T_o - T_w)$, we get

$$\frac{[e_s(T_o) - e_s(T_w)] / (T_o - T_w)}{[e_s(T_o) - e_s(T_w)] / (T_o - T_w) + [e_s(T_w) - e] / (T_o - T_w)} \quad (27)$$

If we simplify the ratio, we get the fraction of total external energy supply going to vaporization $\frac{\Delta}{\Delta + \gamma^*}$. Therefore, the rate of diabatic contribution to latent heat flux density (lE_d) is

$$lE_d = \frac{\Delta}{\Delta + \gamma^*} Q_n \quad (28)$$

The sum of the diabatic and adiabatic contributions provides an estimate of the latent heat flux density

$$lE = lE_d + lE_a = \frac{\Delta}{\Delta + \gamma^*} Q_n + \frac{\rho C_p}{r_H} \left(\frac{e_s(T) - e}{\Delta + \gamma^*} \right) \quad (29)$$

Clearly, this is identical to the Penman-Monteith equation for a leaf (Eq. 13), which was derived using flux gradient concepts. The same steps (Eqs. 14-17) are used to arrive at the Penman-Monteith equation for a canopy (Eq. 18). Remember that

$\frac{e_s(T) - e}{\Delta + \gamma^*}$ is simply an estimate for $T - T_w$ where T_w is the wet-bulb temperature. We are assuming that the surface is wet and the surface temperature is approximately equal to the wet-bulb temperature. If the surface is not 'nearly' wet, then the equation is not valid. We are also assuming that D is a good approximation for $\frac{e_s(T) - e_s(T_w)}{T - T_w}$. This may not be true for very dry air. Perhaps the best use of the

Penman-Monteith equation is to use an independent measure of lE and equation 13 or 18 to investigate physiological differences in stomatal or canopy resistance between plant or crop species. If the assumptions are valid, it can also be used to estimate the maximum evaporation rate for a leaf or canopy.